14.2 Limits and Continuity

Concept: function approaching a limit

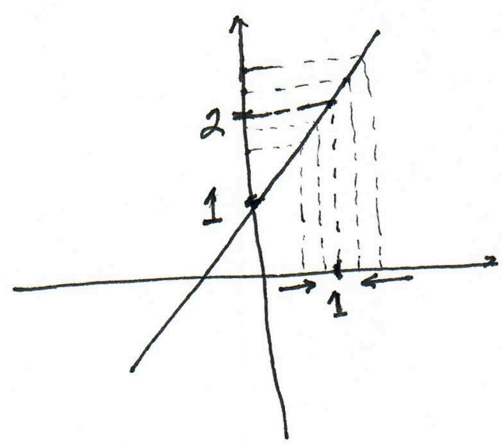
Provisional Definition:

The function f approaches the limit "l" near "a", if we can make $f(x)$ as close as we like to "l" by requiring that x be sufficiently close to, but unequal to, a.

Examples:

1) Consider $f(x) = x+1$

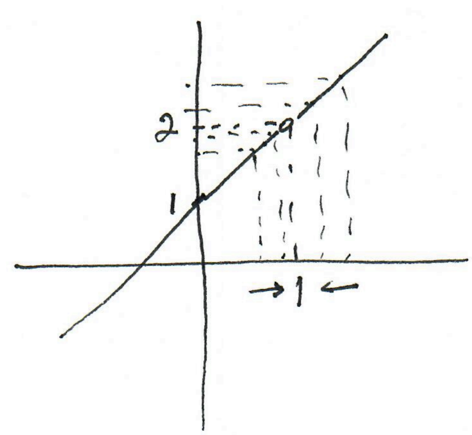
$f(x)$ appraches 2 when x appraches 1.



2) $f(x) = \dfrac{x^2-1}{x-1}, \quad x \neq 1$

$f(x)$ appraaches 2 when x approaches 1

f is not defined at $x = 1$.



Rigurous definition:

The function f approaches the limit "ℓ" near "a"

means:

For every $\varepsilon > 0$ there is some $\delta > 0$ such that,

for all x,

If $0 < |x-a| < \delta$, then $|f(x) - \ell| < \varepsilon$.

Spivak Says:

This definition is so important that

everything (in this class) we do from now depends

on it. ... If necessary memorize it,

like a poem.

Go to example first (Next page)

Negation of the definition: "f does not approach ℓ near a".

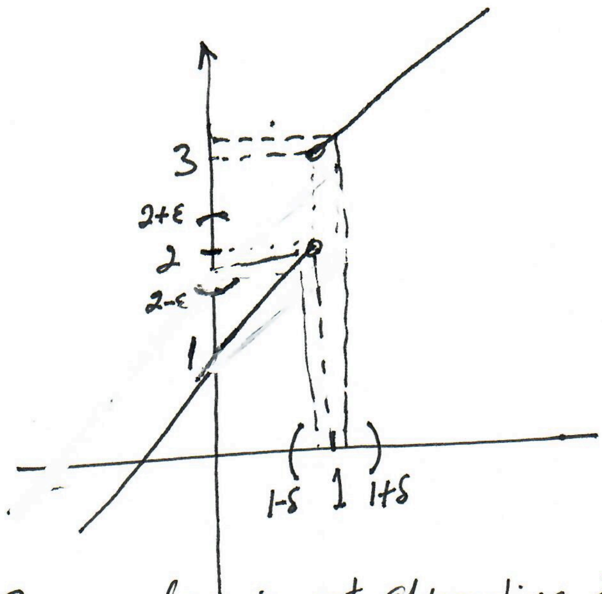
There is some $\varepsilon > 0$ such that for every $\delta > 0$

there is some x which satisfies $0 < |x-a| < \delta$

but not $|f(x) - \ell| < \varepsilon$.

Examples :

$$f(x) = \begin{cases} x+1, & x<1 \\ x+2, & x>1 \end{cases}$$



$f(x)$ is not approaching 2, $f(x)$ is not approaching 3 when x approaches 1

Notation:

f approaches the limit "l" near "a"

$$\lim_{x\to a} f(x) = l.$$

f is not approaching "l" when x approaches a

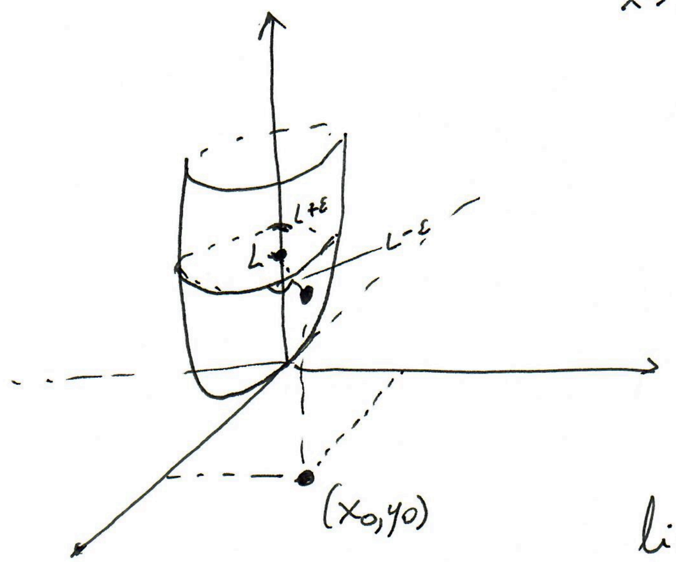
$$\lim_{x\to a} f(x) \neq l.$$

If f is not approaching any l near a

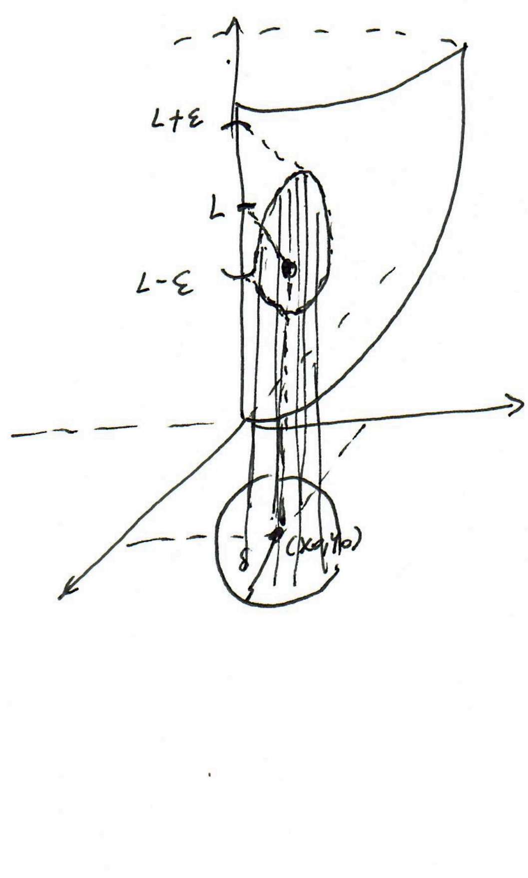
$\lim_{x\to a} f(x)$ does not exist.

Limits for real functions of 2-variables

Show in 1D first for $\lim_{x \to x_0} f(x) = L$



$\lim_{(x,y) \to (x_0,y_0)} f(x,y) = L$



0.4 Definition 4: Limit of functions of two variables

Consider a function $f(x, y)$ defined on an open disc $B_r(a, b)$ centered at the point (a, b) with radius r, $B_r(a, b) = \{(x, y) : \sqrt{(x-a)^2 + (y-b)^2} < r\}$, except possibly at (a, b), then

"the limit of f as (x, y) approaches (a, b) is L" or

$$\lim_{(x,y)\to(a,b)} f(x, y) = L,$$

if for all $\epsilon > 0$, there is a $\delta > 0$ such that
if $(x, y) \in B_r(a, b)$ and $0 < |(x, y) - (a, b)| = \sqrt{(x-a)^2 + (y-b)^2} < \delta$ then $|f(x, y) - L| < \epsilon$

0.5 Theorem 1

If $f(x, y) \to L_1$ as $(x, y) \to (a, b)$ along a path $\mathcal{C}_1$ and $f(x, y) \to L_2$ as $(x, y) \to (a, b)$ along a path $\mathcal{C}_2$, and $L_1 \neq L_2$, then $\lim_{(x,y)\to(a,b)} f(x, y)$ does not exist.

0.6 Definition 5: Partial Derivatives

Let f be a real-valued function of two variables. The partial derivatives of f are the functions defined by

$$f_x(x, y) = \lim_{h\to 0} \frac{f(x+h, y) - f(x, y)}{h}$$

and

$$f_x(x, y) = \lim_{h\to 0} \frac{f(x, y+h) - f(x, y)}{h}$$

provided that these limits exist.

Other notations are

$$f_x(x, y) = \frac{\partial f}{\partial x}(x, y)$$

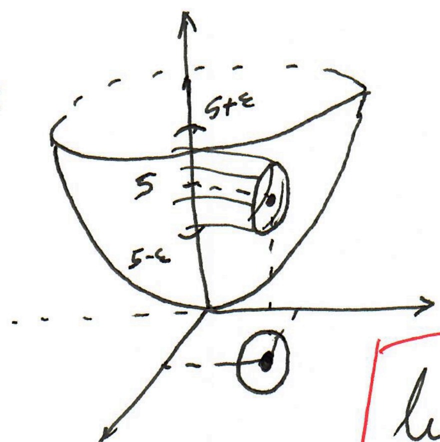
$$f_y(x, y) = \frac{\partial f}{\partial y}(x, y)$$

Limits

Start with

$$\lim_{(x,y)\to(1,2)} x^2+y^2 \stackrel{?}{=}$$

Graph:



then

$$\lim_{(x,y)\to(1,2)} x^2+y^2 = 5.$$

$$\boxed{\lim_{(x,y)\to(a,b)} f(x,y) = \ell}$$

For all $\varepsilon>0$, there is $\delta>0$ s.t. ....

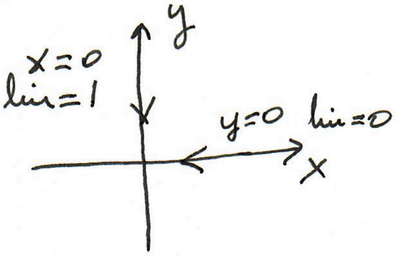
if $0<|(x,y)-(a,b)|<\delta$, then $|f(x,y)-\ell|<\varepsilon$

Go to MAPLE

Study $f(x,y) = \dfrac{yx+y^2}{x^2+y^2}$

Studying limits with MAPLE assistant.

$$\lim_{(x,y)\to(0,0)} \frac{yx+y^2}{x^2+y^2} = \, ?$$



a) Along $y=0$: $\displaystyle\lim_{(x,y)\to(0,0)} \frac{0x+0}{x^2+0}$

$$= \lim_{(x,y)\to(0,0)} \frac{0}{x^2} = 0.$$

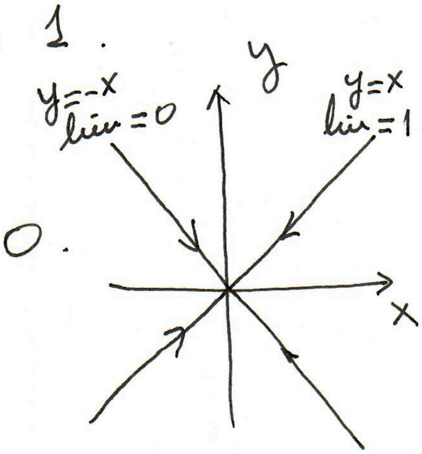
b) Along $x=0$: $\displaystyle\lim_{(x,y)\to(0,0)} \frac{y(0)+y^2}{0+y^2}$

$$= \lim_{(x,y)\to(0,0)} \frac{y^2}{y^2} = \lim_{(x,y)\to(0,0)} 1 = 1$$

c) Along $y=x$: $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2+x^2}{x^2+x^2}$

$$= \lim_{(x,y)\to(0,0)} \frac{2x^2}{2x^2} = \lim_{x\to(0,0)} 1 = 1.$$

d) Along $y=-x$: $\displaystyle\lim_{(x,y)\to(0,0)} \frac{-x^2+x^2}{x^2+y^2} = 0.$



$$\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2} = ?$$

along $y = mx$: $\displaystyle\lim_{(x,y)\to(0,0)} \frac{(mx)\,x^2}{x^4+(mx)^2}$

$$= \lim_{(x,y)\to(0,0)} \frac{mx^3}{x^4+mx^2} =$$

$$= \lim_{(x,y)\to(0,0)} \frac{mx^{\cancel{3}\,1}}{\cancel{x^2}(x^2+m)} = \lim_{(x,y)\to(0,0)} \frac{mx}{x^2+m}$$

$$= \frac{m(0)}{0+m} = \frac{0}{m} = 0$$

Along $y = x^2$:

$$\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2} = \lim_{(x,y)\to(0,0)} \frac{x^4}{x^4+x^4} = \lim \frac{\cancel{x^4}}{2\cancel{x^4}}$$

$$= \lim_{(x,y)\to(0,0)} \frac{1}{2} = \frac{1}{2}$$

<u>Conclusion:</u> This function has a limit along any $(x,y)\to(0,0)$ line $y=mx$ equal to zero.
But, along the parabola $y=x^2$ is $1/2$.
Therefore, there is not limit when $(x,y)\to(0,0)$.

Prove that a given function has a limit using the definition

(Example 6 in book easier to show)

Show
$$\lim_{(x,y)\to(0,0)} \frac{xy^2+y^3}{x^2+y^2} = 0$$

Answ: First, we try to determine if the limit does not exist (which is easier)

Try along lines through the origin

$y = mx$
$$\frac{x m^2x^2 + m^3x^3}{x^2+m^2x^2} = \frac{x^3(m^2+m^3)}{x^2(1+m^2)} =$$

$$= \frac{x(m^2+m^3)}{1+m^2} \xrightarrow[(x,y)\to(0,0)]{} 0.$$

Along parabolas $y = x^2$,
$$\frac{x\,x^4 + x^6}{x^2+x^4} = \frac{x^5(1+x)}{x^2(1+x^2)} =$$

$$= \frac{x^3(1+x)}{1+x^2} \xrightarrow[(x,y)\to(0,0)]{} 0$$

We begin to suspect that the limit exists. and that the limit is equal to zero.

<u>Want to prove:</u> $$\lim_{(x,y)\to(0,0)} \frac{xy^2+y^3}{x^2+y^2} = 0$$

For all $\varepsilon > 0$, there exists $\delta > 0$, such that

if $\quad 0 < \|(x,y)-(0,0)\| = \sqrt{x^2+y^2} < \delta$

then, $\quad \left| f(x,y) - 0 \right| = \left| \frac{xy^2+y^3}{x^2+y^2} \right| < \varepsilon.$

<u>Proof.</u> Notice that

$$|xy^2+y^3| \leq |x|\, y^2 + |y|\, y^2 \leq \sqrt{x^2+y^2}\, y^2 + \sqrt{x^2+y^2}\, y^2$$
$$= 2\sqrt{x^2+y^2}\, y^2 \leq 2\sqrt{x^2+y^2}\,(x^2+y^2)$$

Then, $$\left| \frac{x^2y^2+y^3}{x^2+y^2} \right| \leq 2\sqrt{x^2+y^2}$$

If for a given $\varepsilon > 0$, we select $\delta = \frac{\varepsilon}{2}$

then $$\left| \frac{x^2y^2+y^3}{x^2+y^2} \right| \leq 2\sqrt{x^2+y^2} < 2\,\frac{\varepsilon}{2} = \varepsilon$$

Which is what we want prove.

Another Exercise

Show $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2y^3}{2x^2+y^2} = 0$

Proof. Try first to prove that the limit does not exist.

Not possible to prove it!

Then, we use def. of limit. (ASK students to write what they want to prove)

start with

$$|x^2y^3| \le x^2|y|y^2 \le (x^2+y^2)|y|(x^2+y^2)$$
$$= |y|(x^2+y^2)^2 \le (x^2+y^2)^{1/2}(x^2+y^2)^2 = (x^2+y^2)^{5/2}$$

Then $$\frac{|x^2y^3|}{x^2+y^2} \le (x^2+y^2)^{3/2}$$

Also Notice that

$$\frac{|x^2y^3|}{2x^2+y^2} \le \frac{|x^2y^3|}{x^2+y^2} \le (x^2+y^2)^{3/2}$$

So, if we choose $\underline{\underline{\delta = \varepsilon^{1/3}}}$, and ask for $\sqrt{x^2+y^2} < \delta = \varepsilon^{1/3}$

then, $$\frac{|x^2y^3|}{2x^2+y^2} \le (x^2+y^2)^{3/2} = \left(\sqrt{x^2+y^2}\right)^3 < \delta^3 = (\varepsilon^{1/3})^3 = \varepsilon.$$

which is what we want to prove